

Gaussian potentials facilitate access to quantum Hall states in rotating Bose gases

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Through exact numerical diagonalization for small numbers of atoms, we show that it is possible to access quantum Hall states in harmonically confined Bose gases at rotation frequencies well below the centrifugal limit by applying a repulsive Gaussian potential at the trap center. The main idea is to reduce or eliminate the effective trapping frequency in regions where the particle density is appreciable. The critical rotation frequency required to obtain the bosonic Laughlin state can be fixed at an experimentally accessible value by choosing an applied Gaussian whose amplitude increases linearly with the number of atoms while its width increases as the square root.

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For almost 30 years since its initial observation in a two-dimensional electron gas [1], the quantum Hall (QH) effect has continued to be the subject of intense research. Much recent theoretical work indicates that similar incompressible states should also exist in ultracold trapped atomic gases [2, 3]; the most intriguing possibility would be neutral bosons with tunable two-particle interactions. Many of the proposals exploit the formal equivalence of the Hamiltonian for a two-dimensional interacting electron gas in the presence of an external transverse magnetic field with that describing a neutral interacting gas subject to rotation [4, 5]. Other theoretical approaches make use of internal degrees of freedom to create artificial magnetic fields [6–9].

In addition to their inherent interest, fractional QH states may also be used to perform intrinsically fault-tolerant quantum computing. The topological properties of quasiparticle excitations with non-Abelian statistics could be used to generate a universal set of multi-qubit gates that are protected against the deleterious effects of decoherence [10] by braiding them around each other in specific patterns [11]. The QH state with filling factor $\nu = 5/2$ in electronic systems is widely believed to be described by a Moore-Read wavefunction which possesses the required properties, though recent work has cast some doubt on this [12, 13]. The equivalent state in rotating Bose gases (RBGs) is expected at $\nu = 1$ [5]. A distinct advantage of ultracold atomic gases for topological quantum computing is that quasiparticle creation and manipulation could be as straightforward as shining blue-detuned lasers into the gas and moving them around each other [14].

Unfortunately, the QH effect has not yet been observed in RBGs. Incompressible states result when the number of quantized vortices N_v (the analog of quantized flux in electronic systems) is comparable to the number of particles N , so that the filling factor $\nu \equiv N/N_v \sim 1$. This strongly correlated regime generally requires either very low particle number, or very high rotation frequencies approaching the centrifugal limit of the confining harmonic trap [15]. The first issue can be addressed by

adding a deep one-dimensional optical lattice oriented along the axis of rotation, making an array of disconnected pancake-shaped wells [16], each containing a small number of atoms (conceivably on the order of 10^3), of which all but one or two can be eliminated [17]. The second issue can be partly resolved by applying an approximately quartic potential to the harmonic trap (in practice by adding a repulsive Gaussian potential) [18], though at high rotation frequencies this appears to favor weakly correlated ‘giant vortex’ states [19].

Using exact calculations for small numbers of particles, we demonstrate that the application of a repulsive Gaussian potential with the appropriate parameters can yield bosonic fractional quantum Hall states in RBGs at experimentally accessible rotation frequencies. A simple model agrees closely with the numerical results. To favor the Laughlin state, the combined potentials (harmonic trap, centrifugal term, and external Gaussian) should be flat over the spatial region where the particle density is appreciable. This requires a Gaussian whose amplitude increases linearly with the number of bosons N (with a small coefficient) while the width scales as \sqrt{N} .

The Bose gas at zero temperature is harmonically trapped in an axisymmetric potential with radial and axial trapping frequencies ω and ω_z , respectively, and is rotated around the z -axis at a rate $\tilde{\Omega}$. Imposing tight confinement along the z -axis ($\omega_z \gg \omega$), all motion along this direction is frozen out and the cloud is quasi-two-dimensional, with each particle’s position denoted $\mathbf{r}_i = (\rho_i, \phi_i)$. An external Gaussian potential of the form $V_{\text{ext}} = \tilde{\gamma} e^{-\rho^2/2\tilde{\sigma}^2}$ is included to represent a repulsive blue-detuned laser that is being shined upon the center of the rotating Bose gas, with tunable amplitude $\tilde{\gamma}$ and width $\tilde{\sigma}$. The bosons interact via the usual delta-function pseudopotential with a two-dimensional coupling constant $\tilde{g} = \sqrt{8\pi\hbar\omega\ell^2}a/\ell_z$, where variables $\ell = \sqrt{\hbar/M\omega}$ and $\ell_z = \sqrt{\hbar/M\omega_z}$ are the characteristic oscillator lengths along the radial and axial directions, respectively, and a is the three-dimensional scattering length. In the rotat-

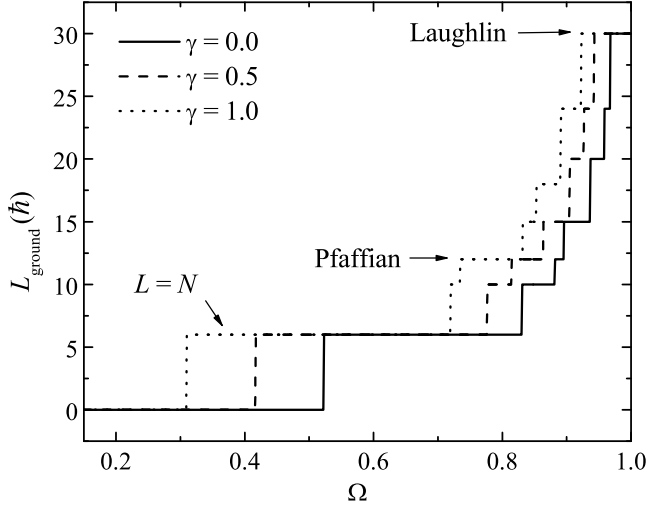


FIG. 1: The yrast lines for the six-particle RBG for different external potential intensities γ . Parameter values are $\sigma = 2.0$ and $g = 0.1$.

ing frame the Hamiltonian is thus

$$H = \sum_{i=1}^N \left[\frac{1}{2M} \left(-i\hbar \nabla_i - M\tilde{\Omega} \times \mathbf{r}_i \right)^2 + \frac{M}{2} \omega_z^2 z_i^2 \right] + \frac{M}{2} \left(\omega^2 - \tilde{\Omega}^2 \right) \rho_i^2 + \tilde{\gamma} e^{-\rho_i^2/2\tilde{\sigma}^2} + \tilde{g} \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j), \quad (1)$$

where N is the number of bosons and M their mass. The energy spectrum of the RBG can then be calculated by exact diagonalization of the Hamiltonian H , for various values of the parameters $\Omega \equiv \tilde{\Omega}/\omega$, $g \equiv \tilde{g}/\hbar\omega\ell^2$, $\gamma \equiv \tilde{\gamma}/\hbar\omega$ and $\sigma \equiv \tilde{\sigma}/\ell$. For more details about the diagonalization process, see Ref. [20].

A Bose condensate is irrotational and thus can only acquire angular momentum through the nucleation of vortices [21]. For small rotation rates, the condensate's ground state has zero angular momentum. As Ω increases and more vortices penetrate the cloud, the system undergoes a series of transitions from states with low to high angular momentum [22], eventually yielding incompressible QH-like states as $\Omega \rightarrow 1$. Fig. 1 shows the corresponding ‘yrast line’ [23] for six particles. Currently, rotation rates in excess of $\Omega = 0.99$ have been achieved in the laboratory [24], though with current experimental conditions the filling factor ν is still an order of magnitude too high to reach the QH limit.

The addition of V_{ext} has a profound effect on the RBG's energy spectrum since a narrow gaussian primarily increases the energy of levels with small angular momentum L , because particles in these states are situated closer to the origin. Consequently, including a repulsive Gaussian favors ground states with higher L , lowering the various critical rotation frequencies. As shown in Fig. 1, increasing γ at constant σ , g , and Ω , changes the ground

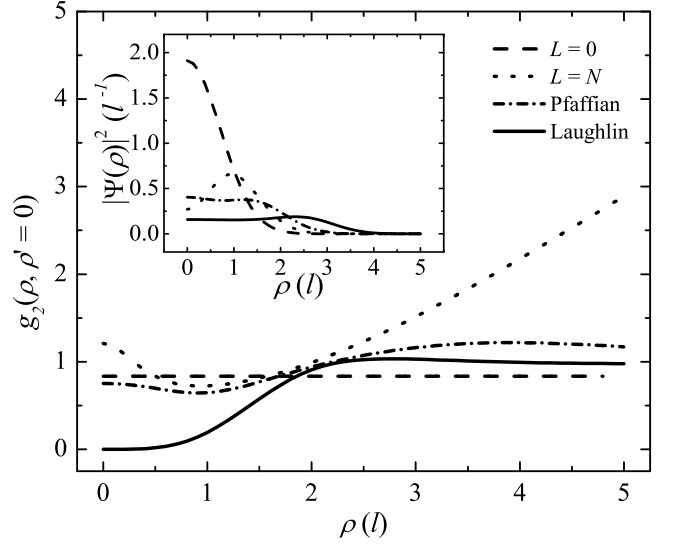


FIG. 2: Second order radial pair correlation function for the $L = 0$, $L = N$, Pfaffian, and Laughlin states of a six-particle RBG, with one particle fixed at the origin ($\rho' = 0$), in the absence of any external potential ($\gamma = 0$). The inset shows the corresponding densities.

state from a Bose condensate with $L = 0$ to the ‘single vortex’ state with $L = N$, eventually to the so-called Pfaffian state at $L = N(N - 2)/2$, and ultimately to the Laughlin state at $L = N(N - 1)$ [22].

While higher angular momentum states may be favored by the presence of V_{ext} , and the resulting particle density might look similar as in its absence, it is not obvious that the strongly correlated nature of these states is preserved. To address this issue one needs to consider higher-order correlation functions. In its normalized form, the pair (also known as the density-density or second-order) correlation function is defined as:

$$g_2(\mathbf{r}, \mathbf{r}') = \frac{\langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) \rangle}{\langle \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \rangle \langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \rangle}, \quad (2)$$

where $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ are the bosonic creation and destruction field operators, respectively, and the expectation value $\langle \dots \rangle$ is taken with respect to the ground state. The density-density correlation function can be interpreted as the probability of a particle being at position \mathbf{r} when another is at position \mathbf{r}' , and has been used to probe for pairing in a superfluid Fermi gas [25, 26] and for density-wave order in bosons confined within deep optical lattices [27–29].

As shown in Fig. 2, the Laughlin state has a particularly distinctive radial pair correlation function $g_2(\rho, \rho')$ (the ϕ dependence is not accessible due to the cylindrical symmetry of our system), which would make its experimental detection unambiguous. The Laughlin wavefunction is an exact solution of the many-body Hamilto-

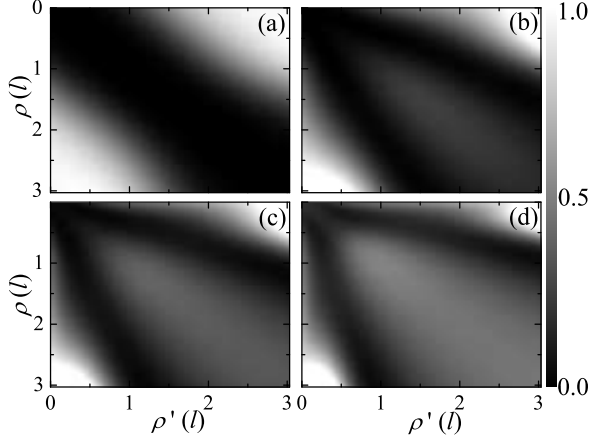


FIG. 3: Radial pair correlation for the five-particle Laughlin state, in the absence of any external potential (a). As the external potential amplitude γ is increased to $\gamma = 0.25$ (b), $\gamma = 0.5$ (c), and $\gamma = 1.0$ (d), the likelihood of overlapping particles increases suggesting that the ground state is no longer the Laughlin. Parameters correspond to $g = 0.1$, and $\sigma = 1.4$.

nian (1) when $\Omega = 1$, and is defined as

$$\Psi_{\text{Laughlin}} = \mathcal{N} \prod_{i < j}^N (\mathbf{z}_i - \mathbf{z}_j)^2 e^{-\sum_i^N |\mathbf{z}_i|^2 / 4\ell_c^2}, \quad (3)$$

where $\mathbf{z}_j = \rho_j e^{i\phi_j}$ is the complex coordinate of particle j , and \mathcal{N} is a normalization constant. The characteristic length $\ell_c = \sqrt{\hbar/m\omega_c}$ is defined in terms of the cyclotron frequency $\omega_c = 2\tilde{\Omega}$, so that $\ell_c \approx \ell/\sqrt{2}$ when $\tilde{\Omega} \sim \omega$. Because of the Jastrow-type prefactor, the value of $g_2(\mathbf{r}, \mathbf{r}')$ is expected to vanish when two particles overlap ($\mathbf{r} = \mathbf{r}'$). In this respect the Laughlin state is most different from the weakly correlated Bose-Einstein condensates at low L , for which $g_2(\mathbf{r}, \mathbf{r}')$ is a constant approaching unity as $N \rightarrow \infty$ for any \mathbf{r} and \mathbf{r}' .

Fig. 3 shows $g_2(\rho, \rho')$ with $\rho, \rho' \leq 3\ell$, for the ground state at $\Omega = 1$ under the influence of different external potential intensities γ and fixed $\sigma = 1.4$. If V_{ext} had no influence on the Laughlin state, the whole diagonal would remain zero. Since this is evidently not the case, it is critical to choose the parameters γ and σ judiciously to best preserve the fundamental characteristics of the Laughlin state, particularly its pair correlations.

For each N the parameters defining V_{ext} were varied in the range $\{0, 20\}$ in their appropriate units, with σ and γ in increments of 0.1 and 0.01, respectively. For each case, the location of the radial pair correlation maximum $g_2(\rho_m, \rho_m)$ along the diagonal was determined, and any parameters for which the maximum exceeded 0.01 were discarded. The values of the parameters γ and σ were minimized, subject to the condition that the RBG ground state is the Laughlin state for some chosen, fixed, value of the rotation frequency $\Omega = \Omega_L < 1$. Results are plotted in Fig. 4 for $\Omega_L = 0.99$ and $g = 0.1$. These indicate that

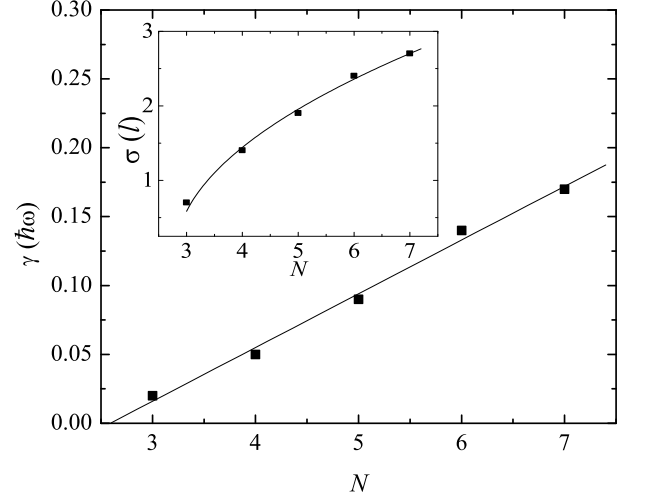


FIG. 4: Optimized magnitude (γ) and width (σ) of the external gaussian potential such that the RBG ground state is the Laughlin state for $\Omega_c = 0.99$ and $g = 0.1$. The lines are best fits, corresponding to $\gamma = (0.039 \pm 0.002)N - (0.10 \pm 0.01)$ and $\sigma = \sqrt{(1.74 \pm 0.06)N - (4.9 \pm 0.3)}$.

a linear increase in γ with N and $\sigma \propto \sqrt{N}$ are sufficient to attain the Laughlin state at $\Omega_L = 0.99$. Extrapolating from the small- N data in Fig. 4, one obtains $\gamma = 39 \pm 2$ and $\sigma = 42 \pm 1$ for $N = 10^3$. Even for $N = 10^4$, the values are not completely unreasonable: $\gamma \approx 400$ and $\sigma \approx 140$. For a choice of $\omega = 100\text{Hz}$, this corresponds to an amplitude of $\tilde{\gamma}/\hbar = 40\text{kHz}$, while optical lattices with depths of 50kHz are readily produced [16].

The qualitative behavior described above was found to be quite general. The numerical calculations were repeated using two and three Landau Levels, and the results were identical to those obtained within the lowest Landau level approximation for interaction strengths $0 \leq g \leq 1$. The optimal values of the Gaussian parameters depend on both g and Ω , but their functional dependence on N is unchanged. When g is increased, it is easier to reach the quantum Hall states [20], and therefore the values of σ and γ are smaller. Similarly, should a lower transition rotation frequency Ω_L be desired keeping g constant, then the Gaussian needs to be higher and wider. For example, for $\Omega_L = 0.95$ the best fits to the data yield $\gamma = (1.05 \pm 0.05)N - (2.3 \pm 0.3)$ and $\sigma = \sqrt{(10.2 \pm 0.5)N - (23 \pm 3)}$ which yields extrapolated values $\gamma = 1050 \pm 50$ and $\sigma = 101 \pm 3$ for $N = 1000$. Evidently, a very low transition frequency for the Laughlin state is not feasible with this technique.

To further validate the pair correlation function as a figure of merit, one requires high overlap $\langle \Psi_{\text{Laugh}} | \Psi \rangle$ between the exact Laughlin state and the one obtained in the presence of the external potential. For all N considered, the overlap integral exceeded 0.99. Plotting the overlap as a function of $1/N$ shows an asymptotic behavior, with a limiting value for large N of 98.5% for the $\Omega_L = 0.99$ data shown in Fig. 4. Thus, for the optimized

values of γ and σ , the state remains the bosonic fractional quantum Hall Laughlin state.

A simple model accounts for the success of applied Gaussians in reducing the critical frequency for the Laughlin state: the combination of the external potentials and the centrifugal barrier should mimic a constant potential in the spatial region where the particle density is appreciable. For large σ , the Gaussian can be expanded in a Taylor series and the radial contribution to the single-particle potential $V(\rho_i)$ in Eq. (1) can be approximated as

$$\frac{V(\rho_i)}{\hbar\omega} \approx \gamma + \frac{1}{2} \left[1 - \Omega^2 - \frac{\gamma}{\sigma^2} \right] \frac{\rho_i^2}{\ell^2} + \frac{\rho_i^4}{8(\sigma\ell)^4} + \mathcal{O}(\rho_i^6). \quad (4)$$

Enforcing that the term in the square brackets vanishes when $\Omega = \Omega_L$, one obtains the relationship between the parameters of the Gaussian: $\gamma = (1 - \Omega_L^2)\sigma^2$. To ensure that the Laughlin state experiences little of the quartic contribution in Eq. (4), one requires that the RMS width of the Laughlin state satisfy $\rho_{\text{RMS}} = \sqrt{\langle \rho^2 \rangle} \lesssim \sigma$ for all N . The spread of Ψ_{Laughlin} can be estimated by noting that the Jastrow prefactor in the definition (3) can be written as a polynomial of order $N(N-1)$ in the variables ρ_i with complex coefficients. Thus we obtain $\Psi_{\text{Laughlin}} \approx \mathcal{N} e^{iN(N-1)\phi} \rho^{N(N-1)} e^{-N\rho^2}$. Straightforward integration yields $\rho_{\text{RMS}} = \sqrt{N-1 + (1/N)} \approx \sqrt{N}$ for $N \gg 1$. Indeed, direct evaluation using (3) for small N gives exactly $\rho_{\text{RMS}} = \sqrt{N}$. Enforcing the condition that $\rho_{\text{rms}} \sim \sigma$, one obtains $\sigma \sim \sqrt{N}$ and $\gamma \sim N$, in agreement with the results shown in Fig. 4.

The best fits to the numerical data are consistent with these scaling relations. For $g = 0.1$ and $\Omega_L = 0.99$, one obtains $\sigma = \sqrt{1.74N - 4.9}$, yielding an optimal value for the amplitude of $\gamma = (1.74N - 4.9)(1 - \Omega_L^2)$, from which the correct slope and y-intercept values are found. Intriguingly, for $\Omega_L = 0.95$, one rather obtains $\sigma = \sqrt{10.2N - 23}$, so that the Gaussian width must be considerably wider than the mean spread of the Laughlin density as the desired rotation frequency is lowered. It is important to note that these choices for the Gaussian parameters facilitate access to not only the Laughlin state, but to all ground states: the effective trap frequency is reduced from $\sqrt{1 - \Omega^2}$ to $\sqrt{\Omega_L^2 - \Omega^2}$, so all transition frequencies are accordingly reduced.

In conclusion, we have demonstrated that it is possible to attain the Bose-Laughlin quantum Hall state, and thus any other quantum Hall state, by applying an external Gaussian potential to a harmonically trapped rotating Bose gas. With experimentally reasonable choices of Gaussian amplitude and width, number of particles, and effective interaction strength, the transition to a Laughlin state can occur for the experimentally accessible rotation rate of 0.99ω or lower. These results suggest that quantum Hall physics in neutral, rotating Bose systems are currently within experimental reach.

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- [1] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
 - [2] N. Regnault and T. Jolicoeur, Phys. Rev. B **69**, 235309 (2004).
 - [3] N. Regnault, C. C. Chang, T. Jolicoeur, and J. K. Jain, J. Phys. B **39**, S89 (2006).
 - [4] N. R. Cooper and N. K. Wilkin, Phys. Rev. B **60**, R16279 (1999).
 - [5] N. Regnault and T. Jolicoeur, Phys. Rev. Lett. **91**, 030402 (2003).
 - [6] D. Jaksch and P. Zoller, New J. Phys. **5**, 56 (2003).
 - [7] E. J. Mueller, Phys. Rev. A **70**, 041603 (2004).
 - [8] A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **94**, 86803 (2005).
 - [9] J. Ruseckas, G. Juzeliunas, P. Ohberg, and M. Fleischhauer, Phys. Rev. Lett. **95**, 010404 (2005).
 - [10] A. Y. Kitaev, preprint quant-ph/9709021 v1 (1997).
 - [11] N. E. Bonesteel, L. Hormozi, G. Zikos, and S. H. Simon, Phys. Rev. Lett. **95**, 140503 (2005).
 - [12] C. Toke and J. K. Jain, Phys. Rev. Lett. **96**, 246805 (2006).
 - [13] C. Toke, N. Regnault, and J. K. Jain, Phys. Rev. Lett. **98**, 036806 (2007).
 - [14] B. Paredes, P. Zoller, and J. I. Cirac, Phys. Rev. A **66**, 033609 (2002).
 - [15] P. Rosenbusch et al., Phys. Rev. Lett. **88**, 250403 (2002).
 - [16] Z. Hadzibabic, S. Stock, B. Battelier, V. Bretin, and J. Dalibard, Phys. Rev. Lett. **93**, 180403 (2004).
 - [17] S. Stock, Z. Hadzibabic, B. Battelier, M. Cheneau, and J. Dalibard, Phys. Rev. Lett. **95**, 190403 (2005).
 - [18] S. Stock, B. Battelier, V. Bretin, Z. Hadzibabic, and J. Dalibard, Laser Phys. Lett. **2**, 275 (2005).
 - [19] A. Aftalion and I. Danaila, Phys. Rev. A **69**, 033608 (2004).
 - [20] A. G. Morris and D. L. Feder, Phys. Rev. A **74**, 033605 (2006).
 - [21] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).
 - [22] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. **84**, 6 (2000).
 - [23] G. F. Bertsch and T. Papenbrock, Phys. Rev. Lett. **83**, 5412 (1999).
 - [24] V. Schweikhard, I. Coddington, P. Engels, V. P. Mosenfelder, and E. A. Cornell, Phys. Rev. Lett. **92**, 040404 (2004).
 - [25] M. Greiner, C. Regal, C. Ticknor, J. Bohn, and D. Jin, Phys. Rev. Lett. **92**, 150405 (2004).
 - [26] C. Lobo, I. Carusotto, S. Giorgini, A. Recati, and S. Stringari, Phys. Rev. Lett. **97**, 100405 (2006).
 - [27] S. Fölling et al., Nature **434**, 481 (2005).
 - [28] L. M. Duan, Phys. Rev. Lett. **96**, 103201 (2006).
 - [29] I. B. Spielman, W. Phillips, and J. Porto, Phys. Rev. Lett. **98**, 080404 (2007).